SHARP INEQUALITIES FOR THE ORNSTEIN-UHLENBECK OPERATOR (JOINT WORK WITH ANDREA CIANCHI AND VÍT MUSIL)

LUBOŠ PICK (CHARLES UNIVERSITY, PRAGUE)

ABSTRACT. The Ornstein–Uhlenbeck operator $\mathcal{L}$, defined by

$$\mathcal{L}u(x) = \Delta u(x) - x \cdot \nabla x,$$

is the natural counterpart of the Laplace operator Δ when the Euclidean space $\mathbb{R}^n$ is replaced by the probablility space $(\mathbb{R}^n, \gamma_n)$, generated by the Gaussian measure

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx.$$

It arises naturally in some problems studied in physics and also in stochastic calculus of variations (the Malliavin calculus). It has applications in many branches of mathematics and physics including financial mathematics and the calculus of variations in infinitely many variables.

For a given function $f \in L^1(\mathbb{R}^n, \gamma_n)$, normalized so that $\int_{\mathbb{R}^n} f d\gamma_n = 0$, there always exists a unique (up to additive constants) solution u to the Ornstein–Uhlenbeck equation

$$\mathcal{L}u = f, \quad \operatorname{med} u = 0$$

in an appropriate weak sense. We study the question of transfer of regularity from f to u and show that an optimal (best possible) such transfer, in a certain sense, is possible. More precisely, given a rearrangement-invariant Banach function space $X(\mathbb{R}^n, \gamma_n)$, the smallest rearrangement-invariant Banach function space $Y(\mathbb{R}^n, \gamma_n)$ can be constructed so that

$$\|u\|_{Y(\mathbb{R}^n,\gamma_n)} \leq C\|f\|_{X(\mathbb{R}^n,\gamma_n)}$$

with a generic positive constant C independent of n and f. We observe certain interesting dissimilarities to the Euclidean case, the prehaps most notable one being that the gain of the degree of integrability inherited by u from f is not always guaranteed. Sharp form of Moser–Adams exponential inequalities is offered as well as the existence of its maximizers.