COMPACT METRIC SPACES AS EXPONENTIABLE METRIC COMPACT HAUSDORFF SPACES

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As it is well-known, many categories of topology are not Cartesian closed in the sense that they do not admit canonical function spaces $[X, Y]$ for all spaces X and Y. This deficiency leaves us with the task to describe those spaces X — called *exponentiable* spaces — which do admit such function space $[X, Y]$ for all Y; put more formally, those spaces X such that the endofunctor $- \times X$ has a right adjoint $[X, -]$. For instance, the exponentiable topological spaces are precisely the core-compact spaces (Day and Kelly 1970), and the exponentiable metric spaces are the totally quasi-convex spaces (Clementino and Hofmann 2006). In this context it might seem rather disappointing that the exponentiable compact Hausdorff spaces are trivial: as shown in Cagliari and Mantovani (1991), a compact Hausdorff space is exponentiable in the category **KH** of compact Hausdorff spaces and continuous maps if and only if it is finite. The situation does not improve if we move to the category **OrdKH** of Nachbin's ordered compact Hausdorff spaces and monotone and continuous maps (Nachbin 1965): it is not difficult to prove that also in this category the exponentiable objects are essentially the finite ones.

Surprisingly, we have more luck considering *metric compact Hausdorff spaces* and appropriate morphisms. These spaces constitute a simultaneous generalisation of ordered compact Hausdorff spaces and of compact metric spaces; in a nutshell, a metric compact Hausdorff space is given by a set equipped with a metric and with a compact Hausdorff topology which is not necessarily induced by the metric but should be compatible in an appropriate sense. The aim of this talk is then to explain the result named in the title of this talk (this part is joint work with Rui Prezado). An important ingredient in the proof is a "Lipschitz version" of Urysohn's Lemma (Matoušková 2000). If time permits, we will also discuss generalisations of Urysohn's Lemma to the context of topometric spaces (Ben Yaacov 2013) (this part is joint work with Marco Abbadini).

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