Sobolev spaces with power weights perturbed by slowly varying functions

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Abstract

We study the compactness of weighted Sobolev embeddings on a bounded domain $\Omega \subset \mathbb{R}^n$

$$\mathrm{id} : E_p^m(\Omega, w) \hookrightarrow L_p(\Omega), \qquad 1 \leq p < \infty, m \in \mathbb{N}$$

in terms of entropy and approximation numbers. Here the space $E_p^m(\Omega, w)$ is the collection of all distributions in the local Sobolev space $W_p^{m,loc}(\Omega)$ such that

$$\|f|E_p^m(\Omega, w)\| := \left(\sum_{|\beta| \leq m} \int_\Omega w(x)^p d(x)^{|\beta|} |\mathrm{D}^\beta f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}} < \infty$$

where $w : \Omega \to (0, \infty)$ is a continuous function and $d(x) := \mathrm{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$, $x \in \Omega$, is the distance function. A special case is the unit ball $\Omega = B = \{x \in \mathbb{R}^n : |x| < 1\}$. Slight modifications lead to the space $E_{p,\psi}^m(B)$ normed by

$$\|f|E_{p,\psi}^m(B)\| := \left(\sum_{|\alpha| = m} \int_\Omega \psi(|x|)^p |x|^{mp} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

where $\psi : (0, 1] \to (0, \infty)$ is *slowly varying* at zero. The particular feature is that the weight $\psi(|x|)^p |x|^{mp} \overset{|x| \to 0}{\longrightarrow} 0$ has a singularity at the origin. It turns out that the growth rate of

$$\lim_{t \to 0} \psi(t) = \infty$$

decisively influences the rate of compactness of the embedding $\mathrm{id} : E_{p,\psi}^m(B) \hookrightarrow L_p(B)$.