Sublocales of the frame of fitted sublocales

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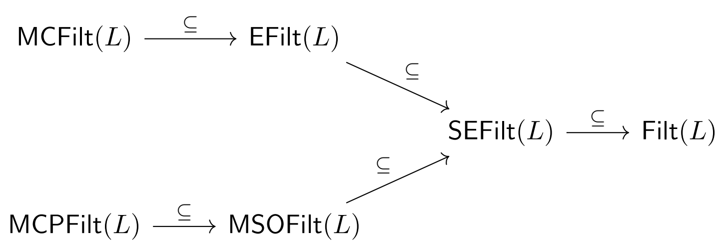
Joint work with Tomáš Jakl

1 Sublocales of the frame of strongly exact filters

When we see a frame L as a pointfree space its *sublocales* are its pointfree subspaces. Sublocales are certain subsets of L, and these form a coframe $\mathsf{S}(L)$ when ordered under set inclusion. We have open sublocales, closed sublocales, and fitted sublocales, which correspond, respectively, to open, closed, and saturated subsets of a space. For a frame L we have an order isomorphism between fitted sublocales and the so-called *strongly exact* filters (see [4]):

$$\mathsf{SEFilt}(L) \cong \mathsf{Fitt}(L)^{op}.$$

Both these posets are frames. I will show in this talk that the frame of strongly exact filters has some distinguished sublocales, which correspond to special subcolocales of $\mathsf{Fitt}(L)$ because of the result above. We have that all the set inclusions in the diagram below (which are easily verifiable) are sublocale inclusions. Here $\mathsf{MCFilt}(L)$ is the frame of meets of *closed* filters, that is, filters of the form $\{x \in L : x \vee a = 1\}$ for some $a \in L$. The frame $\mathsf{EFilt}(L)$ is the frame of *exact* filters, a notion that appears in [4] and is in a certain sense dual to that of strongly exact filter. The frame $\mathsf{MCPFilt}(L)$ is the frame of meets of completely prime filters, and the frame $\mathsf{MSOFilt}(L)$ is the frame of meets of Scott-open filters.



This diagram corresponds to the following diagram of subcolocale inclusions. Here $\mathsf{S}_{\mathfrak{c}}(L)$ is the coframe of joins of closed sublocales (see [5]); $\mathsf{S}_b(L)$ is the coframe of joins of complemented sublocales (whose structure has been recently studied in [1]). $\mathsf{S}_k(L)$ is the coframe of joins of compact sublocales, and $\mathsf{sp}[\mathsf{S}(L)]$ is the coframe of spatial sublocales. Here *fitt* : $\mathsf{S}(L) \to \mathsf{S}(L)$ denotes the *fitting* operator (see [2]), which is the closure coming from $\mathsf{Fitt}(L) \subseteq \mathsf{S}(L)$ seen as a closure system.

$$\begin{array}{ccccc} \mathit{fit}[S_{\mathfrak{c}}(L)] & \xrightarrow{\subseteq} & \mathit{fit}[\mathsf{S}_b(L)] & & \\ & & & \searrow^{\subseteq} & \\ & & & & \mathsf{Fitt}(L) \\ & & & \nearrow_{\subseteq} & \\ \mathit{fit}[\mathsf{sp}[\mathsf{S}(L)]] & \xrightarrow{\subseteq} & \mathit{fit}[\mathsf{S}_k(L)] & & \end{array}$$

2 Polarities: comparing the closure systems $\mathsf{Cl}(L)$ and $\mathsf{Fitt}(L)$

Polarities have been recently explored in [3]. A *polarity* is a complete lattice $\mathsf{Pol}(X, Y, R)$ such that X and Y are sets and $R \subseteq X \times Y$ is a relation, and such that we have two canonical maps $f_X : X \to \mathsf{Pol}(X, Y, R)$ and $f_Y : Y \to \mathsf{Pol}(X, Y, R)$ satisfying a certain universal property. The sublocales of $\mathsf{Fitt}(L)$ above are all instances of polarities. In general, in fact, we have that for a collection $\mathcal{S} \subseteq \mathsf{S}(L)$ of sublocales

$$\mathsf{Pol}(\mathcal{S}, \mathsf{Fitt}(L), \subseteq) = \mathit{fitt}[\mathfrak{J}(\mathcal{S})],$$

where $\mathfrak{J}$ denotes closure under all joins. Therefore the theory of polarities helps us see that the sublocale inclusions above all have universal properties. The theory of polarities also helps us see a symmetry between fitted and closed sublocales. Here for a frame L the expression $\mathfrak{B}(L)$ denotes the Booleanization of L. For a frame L we have:

1. $\mathsf{Pol}(\mathsf{Cl}(L), \mathsf{Op}(L), \subseteq) \cong \mathit{fit}[\mathsf{S}_c(L)] = \mathfrak{B}(\mathsf{Fitt}(L))$;
2. $\mathsf{Pol}(\mathsf{Op}(L), \mathsf{Cl}(L), \subseteq) \cong \mathit{cl}[\mathsf{Op}(L)] = \mathfrak{B}(\mathsf{Cl}(L)) \cong \mathfrak{B}(L)$.

The closure system $\mathsf{Fitt}(L)$ has been recently explored and compared with $\mathsf{Cl}(L)$ in [2]. Since this is work in progress, in the end of the talk I will outline some open questions, especially related to how much more interaction we can find between the theory of polarities and the closure systems within $\mathsf{S}(L)$.

References

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[5] Picado, J., Pultr, A., and Tozzi, A. Joins of closed sublocales. *Houston Journal of Mathematics 45* (01 2019), 21–38.