On a generalized nonlinear heat equation in Besov and Triebel-Lizorkin spaces

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We deal with a generalized nonlinear heat equation

$$\frac{\partial}{\partial t}u(x,t) + (-\Delta_x)^\alpha u(x,t) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} u^2(x,t), \quad x \in \mathbb{R}^n,\ 0 < t < T, \tag{1}$$

$$u(\cdot,0) = u_0(x), \quad x \in \mathbb{R}^n \tag{2}$$

where $0 < T \leq \infty$, $2 \leq n \in \mathbb{N}$, $\alpha \in \mathbb{N}$ and $u(x,t)$ is a scalar function. The case $\alpha = 1$ corresponds to a classical nonlinear heat equation.
In our approach we consider solutions which solve the fixed point problem $T_{u_0}u = u$ whereas the operator T_{u_0} is given as

$$T_{u_0}u(x,t) := W_t^\alpha u_0(x) + \int_0^t W_{t-\tau}^\alpha D u^2(x,\tau)\mathrm{d}\tau, \quad x \in \mathbb{R}^n,\ 0 < t < T$$

in some weighted Lebesgue spaces $L_v((0,T),\, b,\, X)$. Here $X = A^s_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$ is a Banach space of Besov or Triebel-Lizorkin type and $W_t^\alpha \omega$ is defined as

$$W_t^\alpha \omega(x) := \left[\frac{1}{(2\pi)^{n/2}} \left(e^{-t|\xi|^{2\alpha}}\right)^\vee * \omega\right](x), \quad t > 0,\ x \in \mathbb{R}^n$$

for $\omega \in S'(\mathbb{R}^n)$. For this purpose we need the following apriori estimate

$$t^{d/2\alpha} \| W_t^\alpha \omega | A^{s+d}_{p,q}(\mathbb{R}^n)\| \leq c \|\omega | A^s_{p,q}(\mathbb{R}^n)\|, \quad 0 < t \leq 1 \tag{3}$$

with $1 \leq p, q \leq \infty$ ($p < \infty$ in case of F- spaces), $s \in \mathbb{R}$, $d \geq 0$ and $\alpha \in \mathbb{N}$. To prove (3) we use decomposition methods of the spaces $A^s_{p,q}(\mathbb{R}^n)$ by means of wavelets and molecules. Because of the structure of the nonlinearity these spaces have to fulfill certain multiplication properties (multiplication algebra, Hölder inequalities). The initial data u_0 belong to some spaces $A^\sigma_{p,q}(\mathbb{R}^n)$ with $s \geq \sigma$ such that we are in the supercritical case. We ask for solutions of (1), (2) which are additionally strong and stable such that under our conditions the problem is well-posed.

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