conjecture of Alon-Tarsi

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Abstract

A latin square is a $n\times n$ table filled with different symbols, which we may take to be $1,\cdots n$, in a such a way that each symbol occurs exactly once in each row and exactly once in each column.

The sign of a row or column of a latin square,L, is its sign as a permutation of the set $\{1,\cdots,n\}$. A latin square is column(row) even if the product of n column(row) sign is 1. The sign, $s(L)$, of L is the product of $2n$ column and row signs. L is even if $s(L)=1$ and odd if $s(L)=-1$.

In 1986 Alon and Tarsi conjectured the following

Conjecture 1. Let n be an even integer. Then $\Sigma s(L)\neq 0$, where the sum runs over all latin squares L of order n.

We write $CELS(n)$ and $COLS(n)$ to denote the number of column even and column odd latin squares of order n respectively.

In [7,Conjecture 3] was proven that conjecture 1 is equivalent to the following

Conjecture 2. If n is even, then $CELS(n)\neq COLS(n)$.

Let $a_{i,j}=n(i-1)+j$, $i,j=1,\cdots,n$ distinct natural numbers. Consider the following table filled with the $a_{i,j}$

$$\begin{matrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{matrix} \qquad (0.\ 1)$$

Let ζ be the following permutation of S_{n^n}

$$\zeta=(a_{1,1},\ldots,a_{n,1})\cdots(a_{1,i},\ldots,a_{n,i})^i\cdots(a_{1,n-1},\cdots,a_{n,n-1})^{n-1}.$$

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Let H_1 be the subgroup of S_{n^n}

$$H_1 = S_{\{}a_{1,1}, \ldots, a_{1,n}\} \times \cdots \times S_{\{}a_{n,1}, \ldots, a_{n,n}\}. \tag{0. 2}$$

Let λ be the irreducible character associated to the Young diagram (n^n) and let 1_{H_1} be the identity character of H_1. Let

$$\varphi^{H_1}_{\lambda,1_{H_1}}(g) = \frac{1}{|H_1|}\sum_{h\in H_1} \lambda(gh) \ \ , \ \ g \in S_{n^n}$$

be the spherical function associated to λ and 1_{H_1}. We prove that conjecture 2 is true if and only if

$$\varphi^{H_1}_{\lambda,1_{H_1}}(\zeta) \neq 0.$$