About embeddings and minimal action solutions of elliptic equations

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Abstract. I will try to show that the usual connection between Sobolev-type embeddings and the existence of solutions of elliptic equations/systems is much stronger than we may think. In a first part of the talk, we briefly present some multiplicity results for (a) inhomogeneous elliptic equations with a singular Hardy term of the form

$$-\Delta u - \frac{\lambda}{|x|^2}u = |u|^{2^*-2}u + \mu|x|^{\alpha-2}u + f(x),\ \ x \in \Omega\backslash\{0\},\ u \in H_0^1(\Omega),$$

and for (b) Schrödinger-Poisson systems of the form

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^{2^*-2}u + \mu\, h(x)u \\ -\Delta\phi = l(x)u^2 \end{cases},(u,\phi) \in H^1(R^3) \times D^{1,2}(R^3),$$

under adequate assumptions on the parameters. Such problems expose some of the techniques needed when the critical exponent is involved (i.e. noncompact situation) and show explicitly the role of best constants. In a second part, from given Sobolev-type embeddings, I will discuss a scheme that allow us to prove the existence of minimal action solutions of associated elliptic equations and a variational characterisation of its norms. For example, from the 2-dimensional anisotropic Sobolev inequality of the form

$$\int_{R^2} |u|^6\, dx\, dy \le \alpha \left(\int_{R^2} u_x^2\, dx\, dy \right)^2 \int_{R^2} \left| D_x^{-1} u_y \right|^2\, dx\, dy,$$

it is proved the existence of the minimal action solution ϕ of

$$(u_{xx} + |u|^4 u)_x = D_x^{-1} u_{yy}, \quad u \in Y_0(R^2),$$

and that the sharp (smallest) positive constant α is exactly $3\left(\int_{R^2} \phi_x^2\, dx\, dy\right)^{-2}$, where $Y_0(R^2)$ is the closure of $\frac{\partial}{\partial x} C_0^\infty(R^2)$ under the norm

$$\|u\|^2 := \int_{R^2} \left(u_x^2 + \left| D_x^{-1} u_y \right|^2 \right) dx\, dy \quad \text{with } D_x^{-1} h(x,y) := \int_{-\infty}^{x} h(s,y)\, ds.$$

The results mentioned above are joint works with J. Chen or/and L. Huang.