A crash course on fully nonlinear equations

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February 27, 2023

Opening

This is a syllabus for a mini-course on fully nonlinear equations, with an emphasis on viscosity solutions and regularity theory. In what follows, we detail information related to the mini-course.

Description of the course

In the course of the last forty years, the regularity theory for fully nonlinear elliptic equations has developed substantially. Fundamental developments have appeared, whereas important questions remain open as genuine difficulties are yet to be tackled. This mini-course presents a panorama of the theory and points out its building blocks.

The objective of the course is two-fold. First, it aims to equip the audience with the ability to tell a coherent story about the field. Secondly, to provide attendees interested in the topic with the tools necessary to formulate potential open problems. A tentative set of contents is the following.

Preliminaries. Fully nonlinear elliptic equations; definitions and examples. Viscosity solutions: continuous and merely measurable ingredients. General properties and elementary facts; maximum principles and ABP estimates

Regularity theory. The Krylov-Safonov theory. Convexity and the Evans-Krylov theorem. Approximation methods and Caffarelli's regularity theory. Gradient estimates and Święch's estimates. The impossibility of $C^{1,\alpha}$-estimates if integrability falls below the dimension: an example I learned from Boyan Sirakov. Potential estimates, Daskalopoulos-Kuusi-Mingione. The difference quotients and the function-spaces umbrella: facts I learned from Júlio Neves.

Counterexamples and further developments. The counterexamples of Nadirashvili and Vlăduţ; the quaternions and solutions from trialities. 'Knock-knock-dimension 1, 2, 4 and 8: who's there? Division algebras'. Classical regularity for flat solutions. The partial regularity result.

Miscellaneous. Degenerate models; free transmission problems; flipping regularity via the Harnack approach.

Prerequisites and target audience

The course presupposes some knowledge of functional analysis and partial differential equations (a first course at the graduate level should suffice).

The target audience comprises undergraduate students (very) interested in the topic, graduate students in general, and researchers with some curiosity about the field.

Follow-up activities

Potential follow-up activities involve solely students' initiatives. Those include journal clubs on topics stemming from the program or (regular) seminar activities to cover related books.

References for the course

Main reference (for the sessions)

1. Pimentel, E. (2022). Elliptic Regularity Theory by Approximation Methods (London Mathematical Society Lecture Note Series). Cambridge: Cambridge University Press. doi:10.1017/9781009099899

Main references (for the topic)

1. Armstrong, S. N., Silvestre, L., Smart, C. Partial regularity of solutions of fully nonlinear, uniformly elliptic equations. *Comm. Pure Appl. Math.* 65 (2012), no. 8, 1169–1184.

2. Caffarelli, L., Cabré, X. (1995) Fully nonlinear elliptic equations. American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence.

3. Caffarelli, L., Crandall, M. G., Kocan, M., Święch, A. On viscosity solutions of fully nonlinear equations with measurable ingredients. *Comm. Pure Appl. Math.* 49 (1996), no. 4, 365–397.

4. Nadirashvili, N., Tkachev, V., Vlăduţ, S. (2014). Nonlinear elliptic equations and nonassociative algebras. Mathematical Surveys and Monographs, 200. American Mathematical Society, Providence.

5. Savin, O. Small perturbation solutions for elliptic equations. *Comm. Partial Differential Equations* 32 (2007), no. 4-6, 557–578.