

INTERPOLATION POLYNOMIALS IN SEVERAL VARIABLES

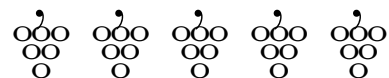
Alain Lascoux



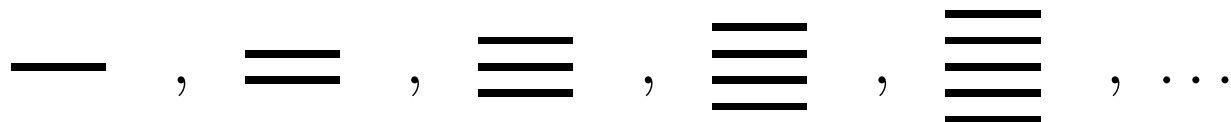
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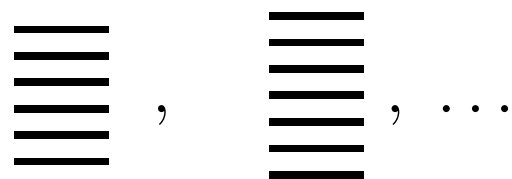
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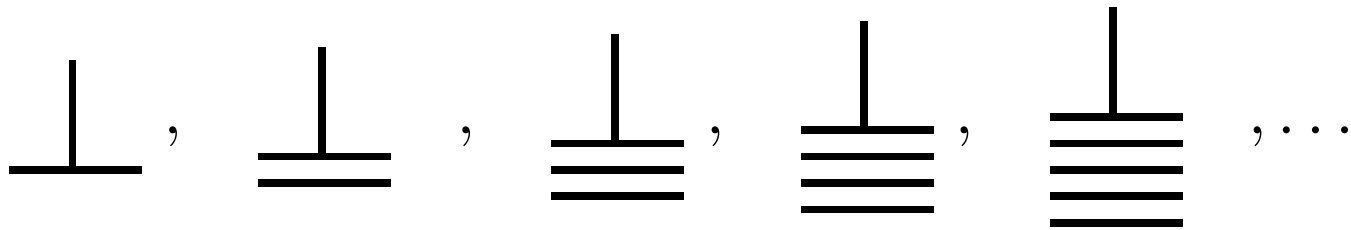
Faced with



everybody continues with



However, the Yigu yanduan of Li Ye prefers :



Modern occidental science is supposed to begin with [Galileo](#), who throws a ball from the Pisa tower and records its position at regular intervals of time :

I, IV, IX, XVI, XXV, XXXVI, ...

The law is a little more complicated, but philosophy comes to the rescue :

Quando, dunque, osservo che una pietra, che discende dall'alto a partire dalla quiete, acquista via nuovi incrementi di velocità, perché non dovrei credere che tali aumenti avvengano secondo la pi semplice e pi ovvia proporzione? Ora, se consideriamo attentamente la cosa, non troveremo nessun aumento o incremento più semplice di quello che aumenta sempre nel medesimo modo quel moto che in tempi eguali, comunque presi, acquista eguali aumenti di velocità.

In short, if it is not the **increment of space** which is uniform, **it must be** the **increment of speed**.

Galileo's method would indeed work for any polynomial law !

```
ACE> [seq( n^3-2*n+3,n=1..10)];  
      [2, 7, 24, 59, 118, 207, 332, 499, 714, 983]
```

The solution of such a problem was already known at the very beginning of astronomy : [compute differences, iterate.](#)

	7	24	59	118	207	332	499
5	17	35	59	89	125	167	
12	18	24	30	36	42		
	6	6	6	6	6		
		0	0	0	0		

From the first diagonal, one can reconstruct (and understand) the original sequence.

However, Chinese mandarins had more serious problems to solve than throwing balls.

For example, in the *Jade Mirror of the four unknowns* one finds

A mandarin recruits soldiers according to cubic numbers. He begins with a 3-foot cube. Then, he increases the side of the cube by one foot each time. Each soldier receives a daily allowance of 250 sapeques. 23400 soldiers have been recruited, and the total expenditure was 23462 silver taels. In how many days were they recruited ?

Notons $f(n)$ le nombre de personnes recrutées le n -ième jour.

n	$f(n)$	différence	2 ^e différence	3 ^e différence	4 ^e différence
0	0				
1	27	$3^3 = 27 (\Delta_1)$			
2	91	$4^3 = 64$	37 (Δ_2)		
3	216	$5^3 = 125$	61	24 (Δ_3)	
4	432	$6^3 = 216$	91	30	6 (Δ_4)
5	775	$7^3 = 343$	127	36	6
...	...				

La quatrième différence étant constante, on aura comme interpolation polynomiale :

$$f(n) = n\Delta_1 + n(n-1)\Delta_2/2! + n(n-1)(n-2)\Delta_3/3! + n(n-1)(n-2)(n-3)\Delta_4/4!$$

Dans le cas présent, $f(n) = 23\,400$.
On obtient $n = 15$ comme une solution de cette équation.

Issue des problèmes d'interpolation en astronomie, l'« art de la différence pour le recrutement » utilise les différences d'ordre 4.

Ici, le « côté de 3 pieds » n'a rien à voir avec une longueur : les hommes étant recrutés suivant les nombres cubiques, le premier jour, $3^3 = 27$ hommes sont recrutés ; on ajoute ensuite un pied, c'est-à-dire que le deuxième jour, $4^3 = 64$ hommes sont recrutés. La méthode de résolution est introduite par la formule : « *L'art dit* : ». Elle correspond à la formule d'interpolation utilisant les différences d'ordre 4.

La formule d'interpolation inventée sous les Sui par Liu Zhuo s'arrêtait à l'ordre 2 ; dans le *Miroir de jade des quatre inconnues*, elle est

However, comets are not likely to appear at regularly spaced times. How to treat them ? This is **Newton** who, while working on the **Principia** found how to transform a **discrete set of data** into an algebraic function :

normalize differences

by

dividing them

by the interval of time

Here is the preceding set of data, minus three observations :

7 ■ 59 ■ ■ 332 499

5 $\frac{52}{2} = 26$ $\frac{273}{3} = 91$ 167

$\frac{21}{3} = 7$ $\frac{65}{5} = 13$ $\frac{76}{4} = 19$

$\frac{6}{6} = 1$ $\frac{6}{6}$

0

According to [Newton](#), the comet position $f(t)$, known at times t_0, t_1, \dots , is :

$$f(t) = f(t_0) + f^\partial(t - t_0) + f^{\partial\partial}(t - t_0)(t - t_1) + \dots$$

obtaining the coefficients $f^\partial, f^{\partial\partial}, \dots$ thanks to the preceding method of dividing differences.

vel semper decrescant. Hoc modo per bisectionem procedi potest usq; dum⁽³⁰⁾ differentiæ quartæ minores sint quam 32.⁽³¹⁾

Possent aliæ hujusmodi regulæ tradi sed mallem rem omnem una regula generali complecti et ostendere quomodo series quævis in loco imperato intercalari⁽³²⁾ possit. Exponatur series per lineas $Ap, Bq, Cr, Ds, Et, Fv, Gw$ &c ad lineam AG perpendiculariter erectas & intervalla terminorum per partes lineæ illius AB, BC, CD, DE &c.⁽³³⁾ Fac $\frac{A-B}{AB} = b,$

$$\frac{B-C}{BC} = b^2, \frac{C-D}{CD} = b^3 \text{ \&c. Item}$$

$$b - b^2 = c, \quad b^2 - b^3 = c^2, \quad b^3 - b^4 = c^3$$

$$\frac{1}{2}AC = c, \quad \frac{1}{2}BD = c^2, \quad \frac{1}{2}CE = c^3$$

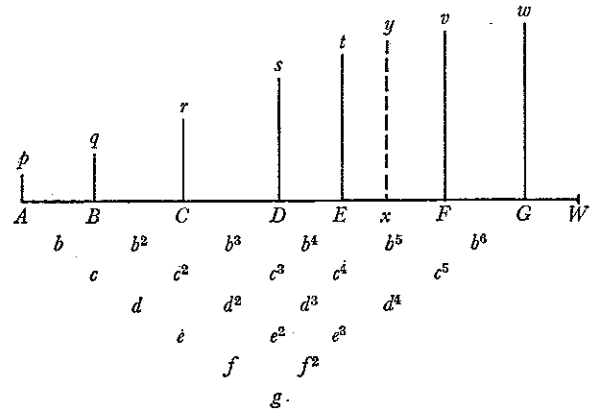
$$\text{\&c. Dein } \frac{c-c^2}{\frac{1}{3}AD} = d, \quad \frac{c^2-c^3}{\frac{1}{3}BE} = d^2,$$

$$\frac{c^3-c^4}{\frac{1}{3}CF} = d^3 \text{ \&[c]. Porro } \frac{d-d^2}{\frac{1}{4}AE} = e.$$

$$\frac{d^2-d^3}{\frac{1}{4}BF} = e^2 \text{ \&c. Tunc } \frac{e-e^2}{\frac{1}{5}AF} = f \text{ \&[c] et sic in sequentibus usq; ad ad finem operis,}$$

dividendo semper differentias primas per intervalla terminorum quorum sunt differentiæ, secundas per dimidium duorum intervallorum quibus respondent, tertias per tertiam partem trium & sic porrò pergendo usq; dum in ultimo loco differentia satis exigua sit.⁽³⁴⁾ Hoc peracto capiantur tum terminorum tum differentiarum primæ A, b, c, d, e, f, g &c. Sit differentiarum illarum numerus $n,$ ⁽³⁵⁾ locus quem intercalare oportet $x,$ terminus intercalaris $xy,$ et regrediendo ab ultima differentia puta g et ab ultimo terminorū ex quibus differentia illa colligebatur puta $G,$ fac $f + \frac{g \times Gx}{n} = p. e + \frac{p \times Fx}{n-1} = q. d - \frac{q \times Ex}{n-2} = r. c - \frac{r \times Dx}{n-3} = s.$

$$b - \frac{s \times Cx}{n-4} = t. A - \frac{t \times Bx}{n-5} = v,$$
⁽³⁶⁾ pergendo semper juxta tenorem progressionis



(30) An unfinished first continuation reads 'præcedentes reg[ulæ applicari possint?]' (the preceding rules [can be applied?]).

(31) Read '21' (compare note (20) above). This paragraph essentially repeats Rule 6 of the preceding piece (see §2: note (34)).

(32) As we would expect (see §2: note (16)) Newton first wrote 'interpolari' (interpolated). The subsequent alteration is bewildering.

(33) Newton has cancelled a following passage, inserting its equivalent below: 'sintq; terminorum differentiæ per intervalla terminorum quibus respondent divisæ[3] primæ quidem $b, b^2, b^3,$ &c, secundæ $c, c^2, c^3,$ &c, tertiæ d, d^2, d^3 &c[3] quartæ e, e^2, e^3 &c & sic ad ultimas. Hoc est.' Observe in sequel that the end-points A, B, C, D, \dots of the lines Ap, Bq, Cr, Ds, \dots are used to denote their magnitude. The end-point W of the line $ABC \dots$ apparently denotes

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always decrease in a regular way. In this manner a bisection procedure may be employed until⁽³⁰⁾ the fourth differences prove to be less than 32.⁽³¹⁾

Other rules of this kind might be presented, but I would prefer to embrace everything in one single general rule and show how any series you wish may be intercalated⁽³²⁾ in any place commanded. Let the series be exhibited by the lines *Ap, Bq, Cr, Ds, Et, Fv, Gw, ...* raised

at right angles to the line *AG*, and the intervals of the terms by the parts *AB, BC, CD, DE ...* of that line.⁽³³⁾ Make

$$\frac{A-B}{AB} = b_1, \quad \frac{B-C}{BC} = b_2, \quad \frac{C-D}{CD} = b_3,$$

$$\dots; \text{ likewise } \frac{b_1-b_2}{\frac{1}{2}AC} = c_1, \quad \frac{b_2-b_3}{\frac{1}{2}BD} = c_2,$$

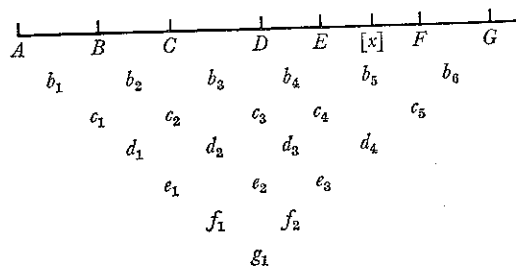
$$\frac{b_3-b_4}{\frac{1}{2}CE} = c_3, \dots; \text{ next } \frac{c_1-c_2}{\frac{1}{3}AD} = d_1, \quad \frac{c_2-c_3}{\frac{1}{3}BE} = d_2, \quad \frac{c_3-c_4}{\frac{1}{3}CF} = d_3, \dots; \text{ further } \frac{d_1-d_2}{\frac{1}{4}AE} = e_1,$$

$$\frac{d_2-d_3}{\frac{1}{4}BF} = e_2, \dots; \text{ then } \frac{e_1-e_2}{\frac{1}{5}AF} = f_1, \dots, \text{ and so on in sequel till the work is finished,}$$

dividing always first differences by the intervals of the terms whose differences they are, second ones by half of the two corresponding intervals, third ones by a third of the three corresponding and so forth until the difference in the final place be slight enough.⁽³⁴⁾ When this is accomplished, take the leading quantities both of the terms and the differences, *A, b₁, c₁, d₁, e₁, f₁, g₁, ...*, and let those differences be *n* in number,⁽³⁵⁾ the place it is required to intercalate call *x*, the term to be intercalated *xy*; then, going backwards from the last difference, say *g₁*, and from the last of the terms, say *G*, from which that difference was gathered,

$$\text{make } f_1 + g_1 \times \frac{Gx}{n} = p, \quad e_1 + p \times \frac{Fx}{n-1} = q, \quad d_1 - q \times \frac{Ex}{n-2} = r, \quad c_1 - r \times \frac{Dx}{n-3} = s,$$

$$b_1 - s \times \frac{Cx}{n-4} = t, \quad A - t \times \frac{Bx}{n-5} = v, \text{ }^{(36)} \text{ proceeding always following the sense of}$$



the last 'known' quantity to be interpolated (*X, Y* and *Z* being inappropriate for the purpose, in context at least).

(34) In continuation Newton first wrote 'puta non major unitate' (suppose not greater than a unit), then replacing it by the unfinished phrase 'puta minor sit quam' (suppose it less than...). A following cancelled amplification at this point reads 'Nam cum *b, b², b³* &c respondeant medijs_[1] interstare supponantur e regione mediorum punctorum inter *A, B, C, D*, &c et *c, c², c³* &c e regione mediorum punctorum inter *b, b², b³*, &c. Distantia terminorum *b* & *b²*_[1] existens summa distantiarum hinc inde a *B*, erit $\frac{1}{2}AB + \frac{1}{2}BC$. [&c]' (For since *b₁, b₂, b₃, ...* correspond to means, let them be supposed to stand between in line with the mid-points between *A, B, C, D, ...*, and *c₁, c₂, c₃, ...* in line with the mid-points between *b₁, b₂, b₃, ...*. The distance of the terms *b₁* and *b₂*, being the sum of their distances either way from *B*, will then be $\frac{1}{2}(AB + BC)$, [and so on].)

(35) Below (and in Newton's diagram) *n* is taken equal to 6.

In more algebraic terms: one starts from a function of x_1, x_2, \dots (the interpolation points), and for each pair x_i, x_{i+1} , one defines an **operator on polynomials** (a **divided difference**) :

$$f \rightarrow f \partial_i := \frac{f(\dots, x_i, x_{i+1}, \dots) - f(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}$$

and one iterates (functions of x_1 are functions of degree 0 in x_2, x_3, \dots , the Newton calculus is indeed a **multivariate calculus** !).

More generally, and more simply, one uses operators T_1, T_2, \dots on polynomials. Each T_i acts on x_i, x_{i+1} only, and commutes with multiplication with symmetric functions in x_i, x_{i+1} . It is therefore sufficient to define its action on a basis, say $1, x_{i+1}$.

Here are 5 examples :

1	1	0	1	1	t
x_{i+1}	x_i	-1	0	$x_i + x_{i+1} - 1$	x_i
	\mathfrak{S}_n	<i>Schubert</i>	<i>Demazure</i>	<i>Grothendieck</i>	<i>Macdonald</i>

I shall only speak of Schubert and Macdonald.

What is the problem ?

- Find linear bases of the ring of polynomials in x_1, \dots, x_n
- Generate their elements
- Expand every polynomial in these bases
- Recover the multiplicative structure

Starting point: monomial, denoted exponentially :

$$\{x^v = x_1^{v_1} \cdots x_n^{v_n} : v \in \mathbb{N}^n\}.$$

Two other bases (triangular, but not with respect to the same order) :

Schubert $\{Y_v : v \in \mathbb{N}^n\}$ & **Macdonald** $\{M_v : v \in \mathbb{N}^n\}$.

One specializes into **spectral vectors** :

$\langle v \rangle$ (components are variables y_j , for **Schubert**),

$\langle v \rangle$ (components are some $t^i q^j$, for **Macdonald**).

Definition: Y_v et M_v are the only polynomials of degree $|v|$ such that

$$Y_v(\langle u \rangle) = 0 \text{ and } M_v(\langle u \rangle) = 0,$$

for each $u : |u| \leq |v|, u \neq v$,

plus normalization conditions

The number of vanishing conditions is equal to the dimension of the space minus 1, no wonder that such polynomials exist ! But this the specific choice of the **spectral vectors** which makes all the beauty and fruitfulness of the theory !

To define the spectral vectors in the **Schubert** case, one needs a bijection between integral vectors and permutations, the **code** of a **permutation**: Given $\sigma \in \mathfrak{S}_N$, its **code** v is the sequence of number of **inversions** due to $\sigma_1, \sigma_2, \dots$, i.e.

$$v_i := \#\{j : j > i \ \& \ \sigma_i > \sigma_j\}$$

Then, one defines

$$\langle v \rangle = [y_{\sigma_1}, y_{\sigma_2}, y_{\sigma_3}, \dots,]$$

i.e. instead of v , one will specialize into the permutation such that v is its code.

To normalize, one chooses the **inversions** of the permutation, i.e. one requires that

$$Y_v(\langle v \rangle) = \mathfrak{m}(v) := \prod_{i < j, \sigma_i > \sigma_j} (y_{\sigma_i} - y_{\sigma_j})$$

For example, $\sigma = [3, 5, 1, 4, 2]$ has code $v = [2, 3, 0, 1, 0]$ and

$$\mathfrak{m}(v) = (y_3 - y_1)(y_3 - y_2)(y_5 - y_1)(y_5 - y_4)(y_5 - y_2)(y_4 - y_2)$$

Supposing known Y_v , with $v_i > v_{i+1}$, one will deduce Y_u , with $u = [v_1, \dots, v_{i-1}, v_{i+1}, v_i - 1, \dots, v_n]$.

Indeed, one writes $Y_v = f + x_i g$, $f, g \in \text{Sym}(x_i, x_{i+1})$.

The equations $Y_v(\langle v \rangle) = \mathfrak{m}(v)$, $Y_v(\langle u \rangle) = 0$, which are $\mathfrak{m}(v) = f(\langle v \rangle) + \langle u \rangle_i g(\langle v \rangle)$, $0 = f(\langle v \rangle) + \langle u \rangle_{i+1} g(\langle v \rangle)$ imply that g be such that $g(\langle v \rangle) = g(\langle u \rangle) = \mathfrak{m}(u)$, and it is not difficult to check that the vanishing conditions are still satisfied.

In summary, $g = Y_v \partial_i$ is the new Schubert polynomial, and **divided differences provide a recursion** between Schubert polynomials.

Initial case: **dominant vectors**, i.e. $v_1 \geq v_2 \geq \cdots \geq v_n$.

One defines

$$Y_v = \prod_{i=1}^n \prod_{j=1}^{v_i} (x_i - y_j)$$

Since this is a product of linear factors, it is not difficult to check the vanishing conditions, together with the normalization condition.

For $v \in \mathbb{N}^n$, let ∂^v be a product of divided differences such that $Y_v \partial^v = Y_{0\dots 0}$. Then, for every other Y_u , either $Y_u \partial^v$ is 0 or equal to Y_w , with $w \neq [0 \dots 0]$. This elementary observation suffices to extend Newton's interpolation to the case of several variables.

Theorem. For every $f \in \mathfrak{Pol}(\mathbf{x}, \mathbf{y})$, one has

$$f(\mathbf{x}) = \sum_{v \in \mathbb{N}^n} f(\mathbf{x}) \partial^v \Big|_{\mathbf{x}=\mathbf{y}} Y_v .$$

Proof: Sufficient to test the statement on the Schubert basis. Since one specializes into $\mathbf{x} = \mathbf{y} = \langle 0 \dots 0 \rangle$, only the term $Y_{0\dots 0}(\langle 0 \dots 0 \rangle) = 1$ survives !

We need new spectral vectors for Macdonald.

When $\lambda \in \mathbb{N}^n$ is dominant,

the **spectral vector** $\langle \lambda \rangle$ is $[t^{n-1}q^{\lambda_1}, \dots, t^0q^{\lambda_n}]$.

Otherwise, if $v = \lambda\sigma$ (σ minimal), one defines

$$\langle v \rangle = \langle \lambda \rangle \sigma.$$

For example, $\langle \mathbf{3, 3, 0} \rangle = [q^3t^2, q^3t^1, q^0t^0]$,

$\langle \mathbf{3, 0, 3} \rangle = [q^3t^2, q^0t^0, q^3t^1]$, $\langle \mathbf{0, 3, 3} \rangle = [q^0t^0, q^3t^2, q^3t^1]$.

Instead of divided differences, one uses the **Hecke algebra** which acts by $1T_i = t$, $x_{i+1}T_i = x_i$.

The operators T_i are not sufficient, one needs an **affine operation**. One takes an **infinite** set of variables x_i , putting $x_{i+rn} = q^r x_i$, with a second parameter q .

Similarly, $v \in \mathbb{N}^n$ must be thought as an infinite vector :

$$v_{i+rn} = v_i + r, r \in \mathbb{Z}.$$

One now has a **translation** τ and its inverse $\bar{\tau} = \tau^{-1}$:

$$\tau : x_i \rightarrow x_{i+1} , v_i \rightarrow v_{i+1}$$

Definition. The **Macdonald polynomial** M_v , $v \in \mathbb{N}^n$ is the **only polynomial** of degree $|v|$ such that

$$\begin{aligned} M_v(\langle u \rangle) &= 0, \quad u \neq v, \quad |u| \leq |v| \\ M_v &= x^v q^{-\sum_i \binom{v_i}{2}} + \dots \end{aligned}$$

Existence and unicity are proved by studying the **compatibility** of vanishing conditions with respect to the **action** of T_i or τ .

One writes $M_v = f + x_{i+1}g$, with $f, g \in \text{Sym}(x_i, x_{i+1})$.
 Since $M_v(\langle v \rangle s_i) = 0$, $M_v(\langle v \rangle) \neq 0$, there is unique
 constant c such that $T_i + c$ exchanges the two
 specializations :

$$M_v \rightarrow F := tf + x_i g + c(f + x_{i+1}g)$$

$$\text{and } F(\langle v \rangle s_i) \neq 0, F(\langle v \rangle) = 0.$$

In final :

$$M_{v s_i} = M_v \left(T_i + \frac{t-1}{\langle v \rangle_{i+1} \langle v \rangle_i^{-1} - 1} \right)$$

The affine operation is no more complicated to follow.
 The polynomial $M_v \bar{\tau}$ inherits all the vanishings of M_v .
 However $v = [v_1, \dots, v_n] \rightarrow v\tau = [v_2, \dots, v_n, v_1+1]$
 increases degree, $M_{v\tau}$ has more vanishings to satisfy, but
 this is provided to by the linear factor.

In final

$$M_{v\tau} = M_v \bar{\tau} (x_n - 1)$$

For example,

$$M_{053}(x_1, x_2, x_3) = M_{205}(x_3/q, x_1, x_2) (x_3 - 1)$$

What kind of applications ? I shall give one to physics, to illustrate that vanishing conditions are not restricted to the mathematical world.

One wants to describe the space of polynomials in degree 6 in x_1, \dots, x_6 , which vanish in all triples

$$[x_i, x_j, x_k] = [t^2, t, 1], \quad i < j < k$$

Answer : The space is 5-dimensional, with basis $M_{210210}, M_{212010}, M_{221010}, M_{212100}, M_{221100}$, specialized in $q = 1/t^3$.

Indeed, one finds that

$$M_{210210} \Big|_{q=1/t^3} = \Delta_t(x_1, x_2, x_3) \Delta_t(x_4, x_5, x_6)$$

with $\Delta_t := \prod_{j>i} (tx_j - x_i)$ the RHS satisfying the required vanishing conditions to be a Macdonald polynomial, though it is **homogeneous** (some care needed, q is not generic!).

The Hecke algebra generate then a 5-dimensional space which is an **irreducible representation** (deforming the Specht representation of the symmetric group corresponding to the partition $[2, 2, 2]$). Because of the specialization of q , the usual vanishing conditions on Macdonald polynomials imply the vanishing on triples $[t^2, t, 1]$, and conversely for this degree.

The physical model which is supposed to be studied is the **XXZ spin chain model with periodic boundary conditions**, or the **Quantum Hall effect**, as well as polynomials solutions of the **Quantum Knizhnik-Zamolodchikov equation**. You can choose ! I prefer the formulation:

“studying the rule $1T_i = t, x_{i+1}T_i = x_i.$ ”